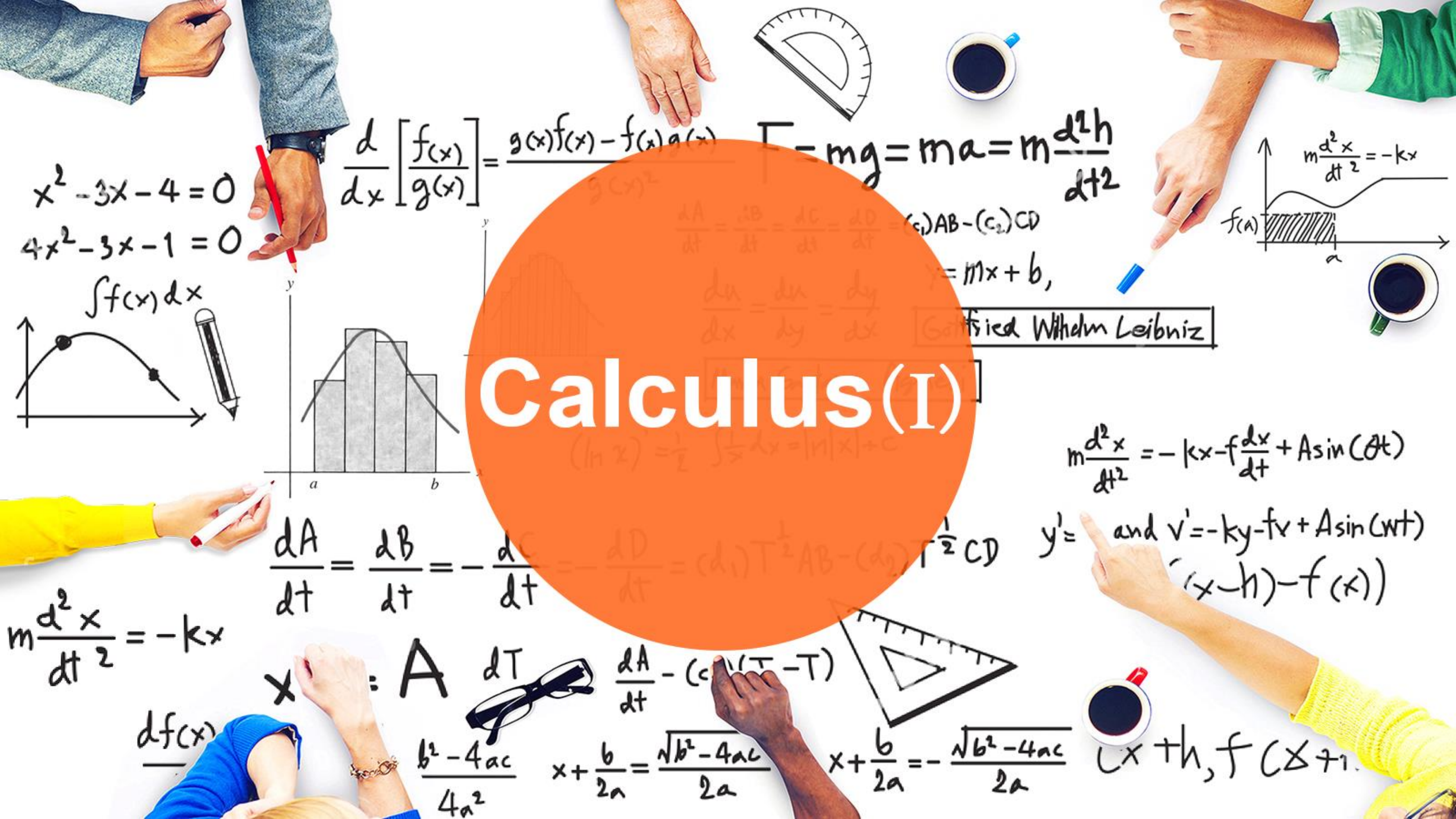
Calculus(I)



Some Trigonometric Integrals

Lecturer: Xue Deng

How to compute these integrations?

$$\int \sin^n x dx = ?$$

$$\int \cos mx \cos nx dx = ?$$

$$\int \cos^n x dx = ?$$

$$\int \sin mx \sin nx dx = ?$$

$$\int \sin mx \cos nx dx = ?$$

$$\int \sin^m x \cos^n x dx = ?$$

Basic Identities

Pythagorean Identities

$$\sin^2 x + \cos^2 x = 1$$

$$1 + \tan^2 x = \sec^2 x$$

$$1 + \cot^2 x = \csc^2 x$$

Half-Angle Identities

$$\sin^2 x = \frac{1 - \cos 2x}{2}$$

$$\cos^2 x = \frac{1 + \cos 2x}{2}$$

Basic Identities

Product Identities

$$(1)\sin mx\cos nx = \frac{1}{2}\left[\sin(m+n)x + \sin(m-n)x\right]$$

$$(2)\sin mx\sin nx = -\frac{1}{2}\left[\cos(m+n)x - \cos(m-n)x\right]$$

$$(3)\cos mx\cos nx = \frac{1}{2}\left[\cos(m+n)x + \cos(m-n)x\right]$$

Example 1

Find $\int \sin^5 x dx$ (n odd)

$$= \int \sin^4 x \sin x dx = -\int \sin^4 x d\cos x$$

$$= -\int (1-\cos^2 x)^2 d\cos x$$

Polynomial of $\cos x$

$$= -\int (1-2\cos^2 x + \cos^4 x) d\cos x$$

$$= -\cos x + \frac{2}{3}\cos^3 x - \frac{1}{5}\cos^5 x + C$$

Example 2

Find $\int \sin^2 x dx$ $(n \text{ even})$

$$= \int \frac{1-\cos 2x}{2} dx$$

Half-Angle Identity

$$= \frac{1}{2}\int (1-\cos 2x) dx$$

$$= \frac{1}{2}x - \frac{1}{4}\sin 2x + C$$

Example 3

Find $\int \sin^2 x \cos^4 x\,dx$ (Both *m* and *n* even)

$$= \int \left(\frac{1-\cos 2x}{2}\right)\left(\frac{1+\cos 2x}{2}\right)^2 dx$$

Half-Angle Identities

$$= \frac{1}{8}\int \left[\frac{1}{2} - \frac{1}{2}\cos 4x + \sin^2 2x \cos 2x\right] dx$$

$$= \frac{1}{8}\left(\frac{1}{2}x - \frac{1}{8}\sin 4x + \frac{1}{6}\sin^3 2x\right) + C$$

Example 4

Find $\int \sin 2x \cos 3x dx$

$$= \frac{1}{2}\int \sin(5x) + \sin(-x) dx$$

Product Identity

$$= \frac{1}{10}\int \sin 5x d5x - \frac{1}{2}\int \sin x dx$$

$$= -\frac{1}{10}\cos 5x + \frac{1}{2}\cos x + C$$

Summary

Pythagorean Identities

$$\sin^2 x + \cos^2 x = 1$$

$$1 + \tan^2 x = \sec^2 x$$

$$1 + \cot^2 x = \csc^2 x$$

Half-Angle Identities

$$\sin^2 x = \frac{1 - \cos 2x}{2}$$

$$\cos^2 x = \frac{1 + \cos 2x}{2}$$

Summary

Product Identities

$$(1)\sin mx\cos nx=\frac{1}{2}\left[\sin(m+n)x+\sin(m-n)x\right]$$

$$(2)\sin mx\sin nx=-\frac{1}{2}\left[\cos(m+n)x-\cos(m-n)x\right]$$

$$(3)\cos mx\cos nx=\frac{1}{2}\left[\cos(m+n)x+\cos(m-n)x\right]$$

Summary

$$\int \sin^m x \cos^n x dx$$

When m, n has an odd number, look for a substitution;

when m, n are both even number, use half-angle identities.

Questions and Answers

Q1: $\int \sin x \cos x dx$

Method1

$$= -\int \cos x(-\sin x)dx$$

substitution

$$= -\int \cos x d\cos x$$

$$= -\frac{1}{2}\cos^2 x + C$$

Questions and Answers

Q1: $\int \sin x \cos x dx$ Method2

$$= \int \sin x(\cos x)dx$$

substitution

$$= \int \sin x d \sin x$$

$$= \frac{1}{2}\sin^2 x + C$$

Questions and Answers

Q2: $\int_{-\pi}^{\pi} \sin mx \sin nx dx$ (m and n are positive integers)

$$= -\frac{1}{2}\int_{-\pi}^{\pi}[\cos(m+n)x - \cos(m-n)x]dx$$

Case1: $m \neq n$

$$= -\frac{1}{2}\left[\frac{1}{m+n}\sin(m+n)x - \frac{1}{m-n}\sin(m-n)x\right]_{-\pi}^{\pi}$$

$$= 0$$

Questions and Answers

Q2: $\int_{-\pi}^{\pi} \sin mx \sin nx dx$ (m and n are positive integers)

$$= -\frac{1}{2}\int_{-\pi}^{\pi}[\cos 2mx - 1]dx$$

Case2: $m = n$

$$= -\frac{1}{2}\left[\frac{1}{2m}\sin 2mx - x\right]_{-\pi}^{\pi}$$

$$= \pi$$

Some Trigonometric Integrals